Complex analysis 1

Frames 1 to 70

Learning outcomes

When you have completed this Programme you will be able to:

- Recognise the transformation equation in the form $w = f(z) = u(x,y) + jv(x,y)$
- Illustrate the image of a point in the complex z-plane under a complex mapping onto the w-plane
- Map a straight line in the z-plane onto the w-plane under the transformation $w = f(z)$
- Identify complex mappings that form translations, magnifications, rotations and their combinations
- Deal with the non-linear transformations $w = z^2$, $w = 1/z$, $w = 1/(z-a)$ and $w = (az+b)/(cz+d)$

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 1 Complex numbers 1, 2 Complex numbers 2 and 3 Hyperbolic functions

1

The foundations of complex numbers and their application to hyperbolic functions were treated fully in Programmes 1, 2 and 3 of *Engineering Mathematics (Fifth Edition)* and these provide valuable revision should you feel it to be necessary before embarking on the new work.

It will be assumed that you are already familiar with the material covered in those previous Programmes and it would be a wise move to work through the relevant Test exercises to refresh your memory on this all-important part of the course.

Functions of a complex variable

For a function of a single real variable $f(x)$ we can construct the graph of the function by plotting points against two mutually perpendicular Cartesian axes, the x-axis and the $f(x)$-axis. For a function of a single complex variable $w = f(z) = u(x, y) + jv(x, y)$ we have four real variables, x, y, u and v. For example if $z = x + jy$ and $f(z) = z^2$ then

$$\begin{aligned} f(z) &= (x + jy)^2 \\ &= x^2 + 2jxy + (jy)^2 \\ &= x^2 - y^2 + 2jxy \end{aligned}$$

and so

$$\begin{aligned} & u(x, y) = x^2 - y^2 \\ \text{and}\quad & v(x, y) = 2xy \end{aligned}$$

We cannot plot the graph of the function $f(z)$ against a single set of axes because to do so we would be required to draw four mutually perpendicular axes which is not possible. Instead, we resort to plotting z-values against x- and y-axes in the complex z-plane and to plotting the corresponding values of $w = f(z)$ against u- and v-axes in the complex w-plane. Accordingly, values of z are plotted on one plane and the corresponding values of $f(z)$ are plotted on another plane. So in our example above for a particular value of z, for example, $z = 4 + j3$

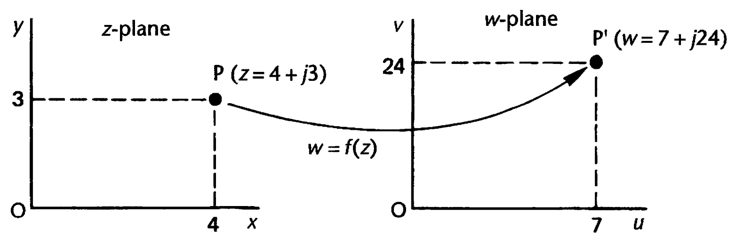
$u = \dots\dots\dots\dots$

$v = \dots\dots\dots\dots$

2

$u = 7 \quad v = 24$

Because with $z = 4 + j3$, $x = 4$ and $y = 3$. Then $u = 16 - 9 = 7$ and $v = 24$.



Therefore, z (where $z = x + jy$) and w (where $w = u + jv$) are two complex variables related by the equation $w = f(z)$.

Any other point in the z-plane will similarly be transformed into a corresponding point in the w-plane, the resulting position P′ depending on

(a) the initial position of P

(b) the relationship $w = f(z)$, called the *transformation equation* or *transformation function*.

Complex mapping

The transformation of P in the z-plane onto P′ in the w-plane is said to be a *mapping* of P onto P′ under the transformation $w = f(z)$ and P′ is sometimes referred to as the *image* of P.

Example 1

Determine the image of the point P, $z = 3 + j2$, on the w-plane under the transformation $w = 3z + 2 - j$.

$$w = u + jv = f(z) = 3z + 2 - j$$
$$= 3(x + jy) + 2 - j$$

so that, for this example,

$u = \ldots\ldots\ldots\ldots;\quad v = \ldots\ldots\ldots\ldots$

3

$u = 3x + 2; \quad v = 3y - 1$

Then the point P ($z = 3 + j2$) transforms onto ....

4

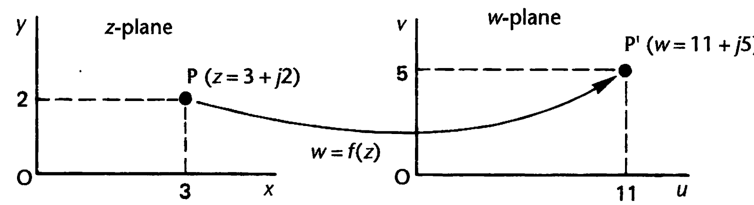
$$\boxed{w = 11 + j5}$$

Because

$z = 3 + j2 \quad \therefore\ x = 3,\ y = 2$

$u = 3x + 2 = 11; \quad v = 3y - 1 = 5; \quad \therefore\ w = 11 + j5$

We can illustrate the transformation thus:



$w = f(z) = 3z + 2 - j$

Here is another.

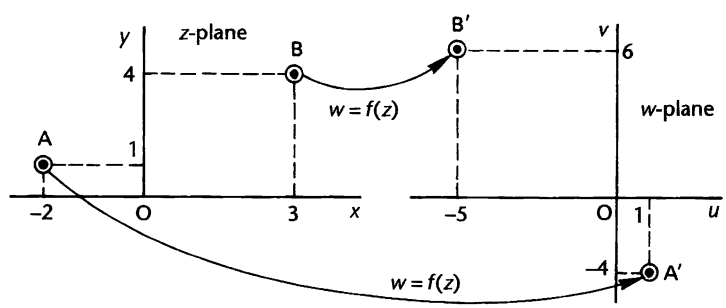
Example 2

Map the points A $(z = -2 + j)$ and B $(z = 3 + j4)$ onto the w-plane under the transformation $w = j2z + 3$ and illustrate the transformation on a diagram.

This is no different from the previous example. Complete the job and check with the next frame.

5

$$\boxed{A'\ (w = 1 - j4); \quad B'\ (w = -5 + j6)}$$



Because

$w = f(z) = j2z + 3 = j2(x + jy) + 3 = (3 - 2y) + j2x$

$w = u + jv \quad \therefore\ u = 3 - 2y; \quad v = 2x$

A: $x = -2,\ y = 1 \quad \therefore\ A'\!: u = 3 - 2 = 1;\ v = -4 \quad \therefore\ A'\!: w = 1 - j4$

B: $x = 3,\ y = 4 \quad \therefore\ B'\!: u = 3 - 8 = -5;\ v = 6 \quad \therefore\ B'\!: w = -5 + j6$

There now follows a short practice exercise. Work all four of the items before you check the results. There is no need to illustrate the transformation in each case.

So move on

6

Exercise

Map the following points in the z-plane onto the w-plane under the transformation $w = f(z)$ stated in each case.

1 $z = 4 - j2$ under $w = j3z + j2$
2 $z = -2 - j$ under $w = jz + 3$
3 $z = 3 + j2$ under $w = (1 + j)z - 2$
4 $z = 2 + j$ under $w = z^2$.

7

1 $w = 6 + j14$	**2** $w = 4 - j2$
3 $w = -1 + j5$	**4** $w = 3 + j4$

That was easy enough. Now let us extend the ideas.

Mapping of a straight line in the *z*-plane onto the *w*-plane under the transformation *w = f(z)*

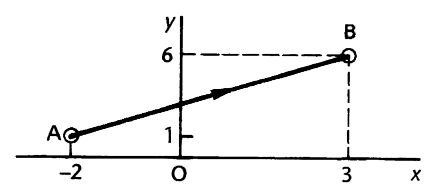
A typical example will show the method.

Example 1

To map the straight line joining A $(-2 + j)$ and B $(3 + j6)$ in the z-plane onto the w-plane when $w = 3 + j2z$.

We first of all map the end points A and B onto the w-plane to obtain A′ and B′ as in the previous cases.

A′: $w = \dots\dots\dots\dots$
B′: $w = \dots\dots\dots\dots$



8

$$\boxed{A'\colon\ w = 1 - j4;\quad B'\colon\ w = -9 + j6}$$

Because

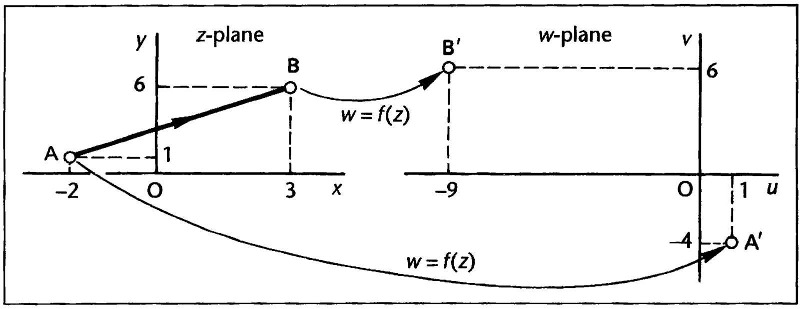
(1) A: $z = -2 + j \qquad w = 3 + j2z$
$\therefore$ A′: $w = 3 + j2(-2 + j) = 3 - j4 - 2 = 1 - j4$

(2) B: $z = 3 + j6$
$\therefore$ B′: $w = 3 + j2(3 + j6) = 3 + j6 - 12 = -9 + j6$

Then, if we illustrate the transformations on a diagram, as before, we get

$\dots\dots\dots\dots$

9



As z moves along the line A to B in the z-plane, we cannot assume that its image in the w-plane travels along a straight line from A′ to B′. As yet, we have no evidence of what the path is. We therefore have to find a general point $w = u + jv$ in the w-plane corresponding to a general point $z = x + jy$ in the z-plane.

$$w = u + jv = f(z) = 3 + j2z$$
$$= \dots\dots\dots\dots\dots\dots$$

10

$$\boxed{w = u + jv = (3 - 2y) + j2x}$$

Because

$$w = 3 + j2(x + jy) = 3 + j2x - 2y = (3 - 2y) + j2x$$
$$\therefore\ u = 3 - 2y \quad \text{and} \quad v = 2x$$

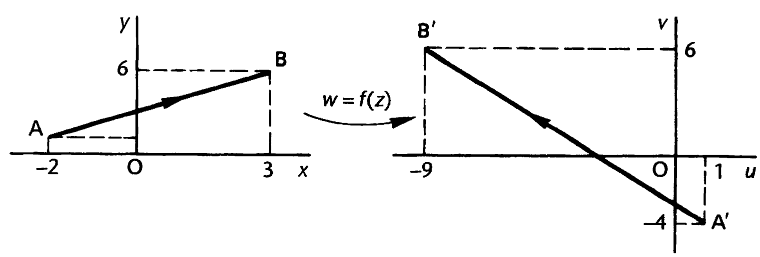
Rearranging these results, we also have $y = \dfrac{3 - u}{2}$; $x = \dfrac{v}{2}$.

Now the Cartesian equation of AB is $y = x + 3$ and substituting from the previous line, we have $\dfrac{3 - u}{2} = \dfrac{v}{2} + 3$ which simplifies to ....

11

$$\boxed{v = -u - 3}$$

which is the equation of a straight line, so, in this case, the path joining A′ and B′ *is* in fact a straight line.



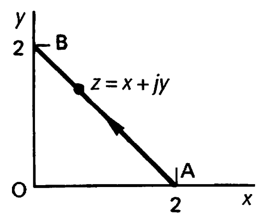
Note that it is useful to attach arrow heads to show the corresponding direction of progression in the transformation.

On to the next

12

Example 2

If $w = z^2$, find the path traced out by w as z moves along the straight line joining A $(2 + j0)$ and B $(0 + j2)$.



Cartesian equation of AB is

$y = 2 - x$

First we transform the two end points A and B onto A′ and B′ in the w-plane.

A′: ....; B′: ....

13

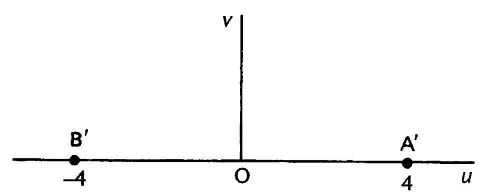
$$\boxed{\text{A}': w = 4 + j0; \quad \text{B}': w = -4 + j0}$$

Because

$w = z^2$ A: $z = 2$ $\therefore$ A′: $w = 2^2 = 4$

B: $z = j2$ $\therefore$ B′: $w = (j2)^2 = -4$

So we have



Now we have to find the path from A′ to B′.

The Cartesian equation of AB in the z-plane is $y = 2 - x$.

Also $w = z^2 = (x + jy)^2 = (x^2 - y^2) + j2xy$

$\therefore\ u = x^2 - y^2$ and $v = 2xy$

Substituting $y = 2 - x$ in these results we can express u and v in terms of x.

$u =$ ....; $v =$ ....

14

$$\boxed{u = 4x - 4; \quad v = 4x - 2x^2}$$

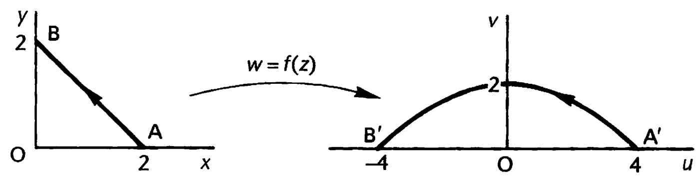
So, from the first of these $\quad x = \dfrac{u+4}{4}$

Substituting in the second $\quad v = 4\left(\dfrac{u+4}{4}\right) - 2\left(\dfrac{u+4}{4}\right)^2$

$$= u + 4 - \frac{1}{8}(u^2 + 8u + 16)$$

$$= -\frac{1}{8}(u^2 - 16)$$

Therefore the path is $v = -\dfrac{1}{8}(u^2 - 16)$ which is a parabola for which at $u = 0$, $v = 2$.

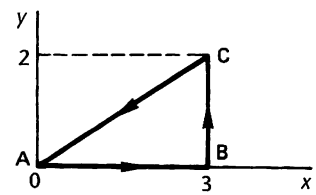


Note that a straight line in the z-plane does not always map onto a straight line in the w-plane. It depends on the particular transformation equation $w = f(z)$.

If the transformation is a *linear equation*, $w = f(z) = az + b$, where a and b may themselves be real or complex, then a straight line in the z-plane maps onto a corresponding straight line in the w-plane.

Example 3

A triangle in the z-plane has vertices at A ($z = 0$), B ($z = 3$) and C ($z = 3 + j2$). Determine the image of this triangle in the w-plane under the transformation equation $w = (2 + j)z$.



$$w = u + jv = f(z) = (2+j)z = (2+j)(x+jy) = (2x - y) + j(2y + x)$$

$$\therefore\; u = 2x - y; \qquad v = 2y + x$$

We now transform each vertex in turn onto the w-plane to determine A′, B′ and C′.

These are A′: ....; B′: ....; C′: ....

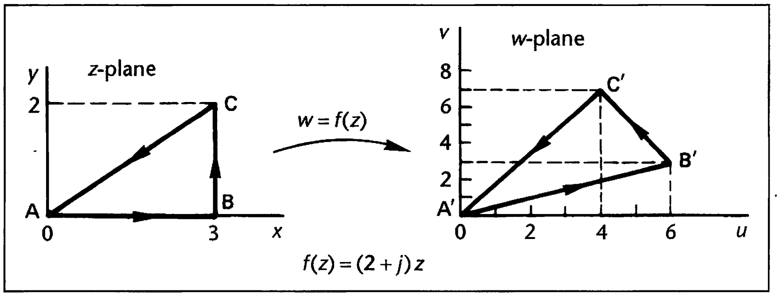
15

$$\text{A}': w = 0; \quad \text{B}': w = 6 + j3; \quad \text{C}': w = 4 + j7$$

The transformation is linear (of the form $w = az$) so A′B′, B′C′ and C′A′ are straight lines and the transformation can be illustrated in the diagram

. . . .

16



All very straightforward. Let us now take a more detailed look at linear transformations.

Types of transformation of the form *w* = *az* + *b*

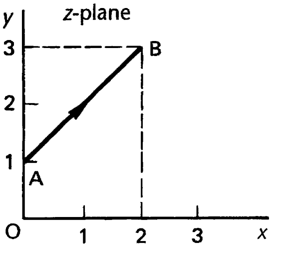
where the constants a and b may be real or complex.

1 *Translation*

Let $a = 1$ and $b = 2 - j$ i.e. $w = z + (2 - j)$.

If we apply this to the straight line joining A $(0 + j)$ and B $(2 + j3)$ in the z-plane, then

$$w = x + jy + 2 - j$$
$$= (x + 2) + j(y - 1)$$

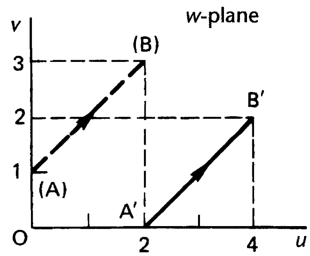


so the corresponding end points A′ and B′ in the w-plane are

A′: . . . . ; B′: . . . .

17

$$\boxed{A': w = 2; \quad B': w = 4 + j2}$$



The transformed line A′B′ is then as shown. The broken line (A)(B) indicates the position of the original line AB in the z-plane.

Note that the whole line AB has moved two units to the right and one unit downwards, while retaining its original magnitude (length) and direction.

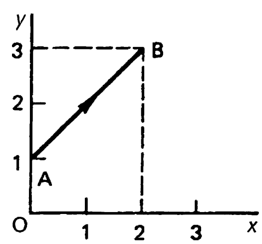
Such a transformation is called a *translation* and occurs whenever the transformation equation is of the form $w = z + b$. The degree of translation is given by the value of b – in this case $(2 - j)$, i.e. 2 units along the positive real axis and 1 unit in the direction of the negative imaginary axis.

On to the next frame

18

2 *Magnification*

Consider now $w = az + b$ where $b = 0$ and a is real, e.g. $w = 2z$.



Applying the transformation to the same line AB as before, we have

$$w = u + jv = 2z = 2(x + jy)$$
$$\therefore \; u = 2x \quad \text{and} \quad v = 2y$$

Transforming the end points A $(0 + j)$ and B $(2 + j3)$ onto A′ and B′ in the w-plane, we have

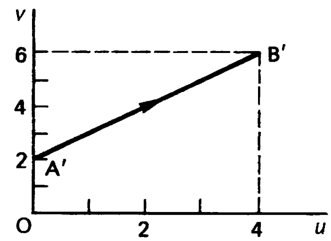
A′: ....; B′: ....

and the w-plane diagram becomes

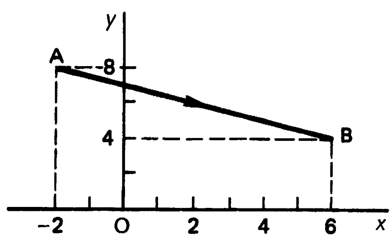
....

19

$$\boxed{A': w = j2; \quad B': w = 4 + j6}$$



Note that (a) all distances in the z-plane are magnified by a factor 2, and (b) the direction of A′B′ is that of AB unchanged. Any such transformation $w = az$ where a is real, is said to be a *magnification* by the factor a.

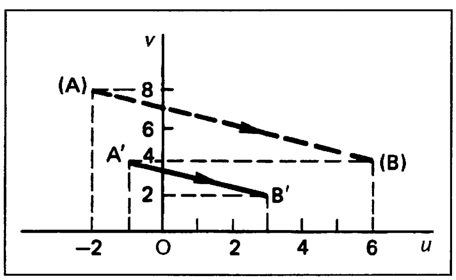


So, if we apply the transformation $w = z/2$ to AB shown here, we can map AB onto A′B′ in the w-plane and obtain

....

Sketch the result

20

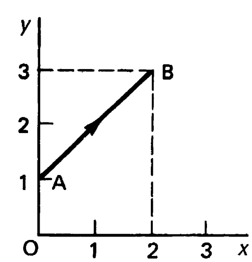


3 *Rotation*

Consider next $w = az + b$ with $b = 0$ and a complex,

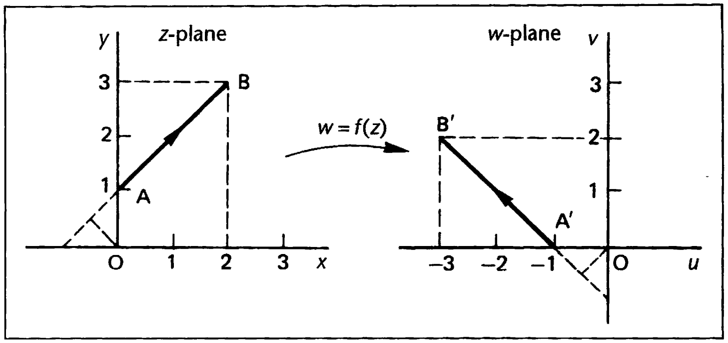
e.g. $w = jz$.

$$w = u + jv = jz$$
$$= j(x + jy)$$
$$= -y + jx$$



Transforming the end points as usual, we can sketch the original line AB and the mapping A′B′, which gives ....

21



A′ is the point $w = -1 + j0$; B′ is the point $w = -3 + j2$

Note $AB = 2\sqrt{2}$ $A'B' = 2\sqrt{2}$

Slope of $AB = m = 1$ Slope of $A'B' = m_1 = -1$

$mm_1 = 1(-1) = -1$

Therefore in transformation by $w = jz$, AB retains its original length but is rotated about the origin, in this case through 90° in a positive (anticlockwise) direction.

Some degree of rotation always occurs when the transformation equation is of the form $w = az + b$ with a complex.

Move on to the next frame

22

4 *Combined magnification and rotation*

If $w = (a + jb)z$, the effect of transformation is

(a) magnification $|a + jb| = \sqrt{a^2 + b^2}$

(b) rotation anticlockwise through $\arg(a + jb)$, i.e. $\arctan \dfrac{b}{a}$.

Let us see this with an example.

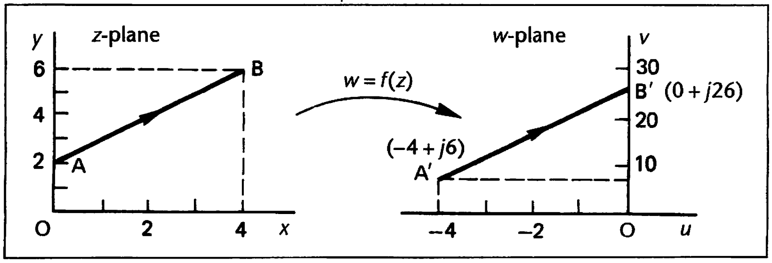
Example

Map the straight line joining A $(0 + j2)$ and B $(4 + j6)$ in the z-plane onto the w-plane under the transformation $w = (3 + j2)z$.

The working is just as before. Draw the z-plane and w-plane diagrams, which give

....

23



$$w = (3 + j2)z$$

$$\therefore\; u + jv = (3 + j2)(x + jy) = (3x - 2y) + j(2x + 3y)$$

$$\therefore\; u = 3x - 2y \quad \text{and} \quad v = 2x + 3y$$

$$\text{A: } z = 0 + j2, \text{ i.e. } x = 0,\; y = 2$$

$$\therefore\; \text{A}': u = -4,\; v = 6 \quad \therefore\; \text{A}': (-4 + j6)$$

$$\text{B: } z = 4 + j6, \text{ i.e. } x = 4,\; y = 6$$

$$\therefore\; \text{B}': u = 0,\; v = 26 \quad \therefore\; \text{B}': (0 + j26)$$

By a simple application of Pythagoras, we can now calculate the lengths of AB and A′B′, and then determine the magnification factor (A′B′)/(AB).

AB = ...; A′B′ = ...; magnification = ...

24

$$\boxed{\text{AB} = 4\sqrt{2};\quad \text{A}'\text{B}' = 4\sqrt{26};\quad \text{mag} = \sqrt{13}}$$

Because

$$\text{AB} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$$

$$\text{A}'\text{B}' = \sqrt{16 + 400} = \sqrt{416} = 4\sqrt{26}$$

$$\therefore\; \text{magnification} = \frac{4\sqrt{26}}{4\sqrt{2}} = \sqrt{13}$$

Also $|a + jb| = |3 + j2| = \sqrt{9 + 4} = \sqrt{13} \quad \therefore\; \text{mag} = |a + jb|$

Now let us check the rotation.

For AB $\quad \tan\theta_1 = 1 \quad \therefore\; \theta_1 = 45^\circ = 0{\cdot}7854$ radians

For A′B′ $\quad \tan\theta_2 = 5 \quad \therefore\; \theta_2 = 78^\circ\, 41' = 1{\cdot}3733$ radians

$$\therefore\; \text{rotation} = \theta_2 - \theta_1 = 1{\cdot}3733 - 0{\cdot}7854 = 0{\cdot}5879$$

i.e. rotation = 0·5879 radians

Also $\arg(a + jb) = \arg(3 + j2) = \ldots\ldots\ldots\ldots$

25

$\boxed{0{\cdot}5879 \text{ radians}}$

Because $\arg(3+j2) = \arctan\frac{2}{3} = 33^\circ\ 41' = 0{\cdot}5879$ radians.

So, in transformation $w = (a+jb)z = (3+j2)z$

(a) AB is magnified by $|a+jb|$, i.e. $\sqrt{13}$

(b) AB is rotated anticlockwise through $\arg(a+jb)$, i.e. $\arg(3+j2)$ i.e. $0{\cdot}5879$ radians.

5 *Combined magnification, rotation and translation*

The work we have just done can be extended to include all three effects of transformation.

In general, a transformation equation $w = az + b$, where a and b are each real or complex, results in

magnification $|a|$; rotation $\arg a$; translation b

Therefore, if $w = (3+j)z + 2 - j$

magnification = ...; rotation = ...;

translation = ...

26

mag $= \sqrt{10} = 3{\cdot}162$; rotation $= 18^\circ\ 26' = 0{\cdot}3218$ radians; translation = 2 units to right, 1 unit downwards

Because

(a) magnification $= |3+j| = \sqrt{9+1} = \sqrt{10} = 3{\cdot}162$

(b) rotation $= \arg(3+j) = \arctan\dfrac{1}{3} = 18^\circ\ 26' = 0{\cdot}3218$ radians

(c) translation $= 2 - j$, i.e. 2 to the right, 1 downwards.

Let us work through an example in detail.

Example 1

The straight line joining A $(-2-j3)$ and B $(3+j)$ in the z-plane is subjected to the linear transformation equation

$w = (1+j2)z + 3 - j4$

Illustrate the mapping onto the w-plane and state the resulting magnification, rotation and translation involved.

The first part is just like examples we have already done. So,

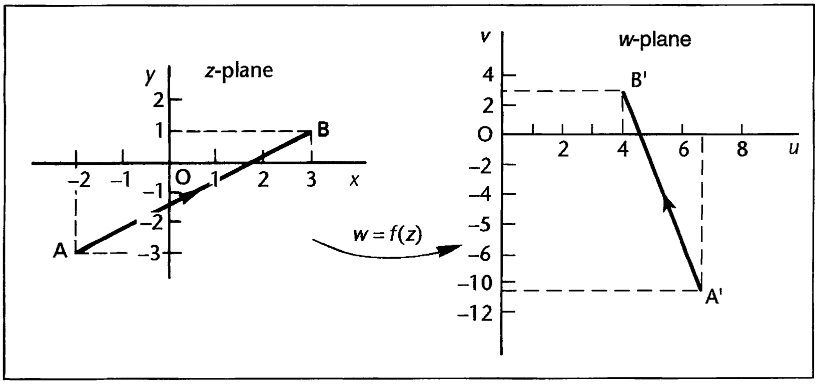
(a) transform the end points A and B onto A′ and B′ in the w-plane

(b) join A′ and B′ with a straight line, since AB is a straight line and the transformation equation is linear.

That can be done without trouble, the final diagram being

....

27



Check the working. $w = (1 + j2)z + 3 - j4$

A: $z = x + jy$

$= -2 - j3$

A′: $w = u + jv = (1 + j2)(-2 - j3) + 3 - j4$

$= -2 - j7 + 6 + 3 - j4$

$= 7 - j11$

B: $z = x + jy$

$= 3 + j$

B′: $w = u + jv = (1 + j2)(3 + j) + 3 - j4$

$= 3 + j7 - 2 + 3 - j4$

$= 4 + j3$

Now for the second part of the problem, we have to state the magnification, rotation and translation when $w = (1 + j2)z + 3 - j4$. We remember that the 'tailpiece', i.e. $3 - j4$, independent of z, represents the

....

28

translation

So, for the moment, we concentrate on $w = (1 + j2)z$, which determines the magnification and rotation. This tells us that

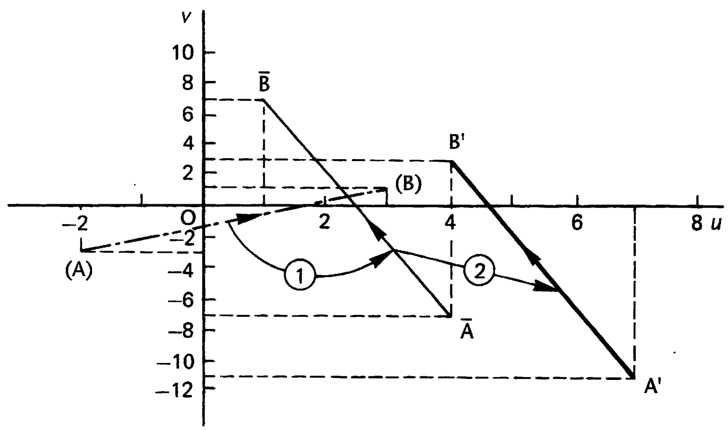
magnification = ...
rotation = ...

29

$$\text{mag} = |a| = |1 + j2| = \sqrt{1+4} = \sqrt{5} = 2{\cdot}236$$
$$\text{rotation} = \arg a = \arctan \tfrac{2}{1} = 63^\circ\ 26' = 1{\cdot}107 \text{ radians}$$

The translation is given by $(3 - j4)$, i.e. 3 units to the right, 4 units downwards.

We can in fact see the intermediate steps if we deal first with the transformation $w = (1 + j2)z$ and subsequently with the translation $w = 3 - j4$.



Under $w = (1 + j2)z$, A and B map onto $\overline{\text{A}}$ and $\overline{\text{B}}$ where $\overline{\text{A}}$ is $w = 4 - j7$ and $\overline{\text{B}}$ is $w = 1 + j7$.

Then the translation $w = 3 - j4$ moves all points 3 units to the right and 4 units downwards, so that $\overline{\text{A}}$ and $\overline{\text{B}}$ now map onto A′ and B′ where A′ is $w = 7 - j11$ and B′ is $w = 4 + j3$.

Normally, there is no need to analyse the transformation into intermediate steps.

Now for –

Example 2

Map the straight line joining A $(1 + j2)$ and B $(4 + j)$ in the z-plane onto the w-plane using the transformation equation

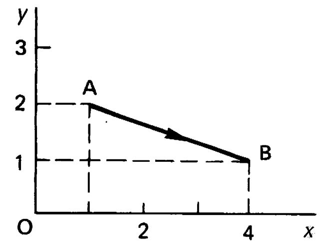
$$w = (2 - j3)z - 4 + j5$$

and state the magnification, rotation and translation involved.

There are no snags. Complete the working and check with the next frame.

30

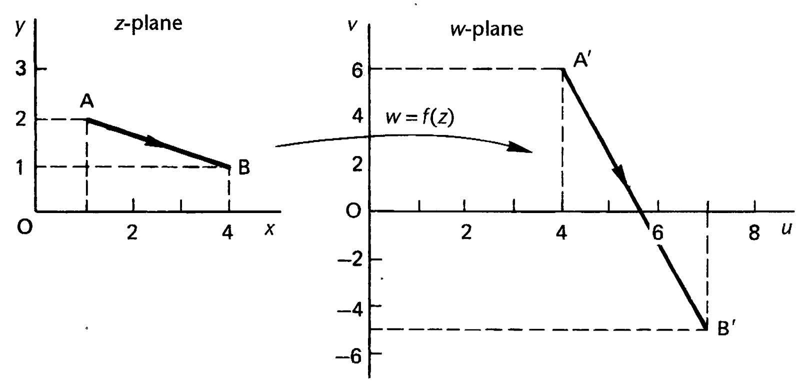
Here is the complete working.



$w = (2 - j3)z - 4 + j5$
A: $z = 1 + j2$
B: $z = 4 + j$

A: $z = 1 + j2$
A′: $w = (2 - j3)(1 + j2) - 4 + j5 = 2 + j + 6 - 4 + j5 = 4 + j6$
B: $z = 4 + j$
B′: $w = (2 - j3)(4 + j) - 4 + j5 = 8 - j10 + 3 - 4 + j5 = 7 - j5$

So we have



Also we have

(a) magnification $= |2 - j3| = \sqrt{4+9} = \sqrt{13} = 3{\cdot}606$
(b) rotation $= \arg(2 - j3) = \arctan(\frac{-3}{2}) = -56^\circ\ 19'$
$= 0{\cdot}9828$ radians clockwise
(c) translation $= -4 + j5$ i.e. 4 units to left, 5 units upwards

All very straightforward. Before we move on, here is a short revision exercise.

Exercise

Calculate (a) the magnification, (b) the rotation, (c) the translation involved in each of the following transformations.

1 $w = (1 - j2)z + 2 - j3$
2 $w = (4 + j3)z - 2 + j5$
3 $w = (2 - j3)z - 1 - j$
4 $w = (j - 4)z + j2 - 3$
5 $w = j2z + 4 - j$
6 $w = (5 + j2)z + j(j3 - 4)$.

Complete all six and then check the results with the next frame.

31

Results:

1 $w = (1 - j2)z + 2 - j3$

(a) magnitude $= |1 - j2| = \sqrt{1+4} = \sqrt{5} = 2{\cdot}236$

(b) rotation $= \arg(1 - j2) = \arctan(-2) = -63° \, 26'$

$= 1{\cdot}107$ radians clockwise

(c) translation $= 2 - j3$, i.e. 2 units to right, 3 units downwards.

The others are done in the same way and give the following results.

No.	Magnitude	Rotation (rad)	Translation
2	5	0·6435 ac	2L, 5U
3	3·606	0·9828 c	1L, 1D
4	4·123	0·2450 c	3L, 2U
5	2	1·5708 ac	4R, 1D
6	5·385	0·3805 ac	3L, 4D

Now let us start a new section, so on to the next frame

Non-linear transformations

32

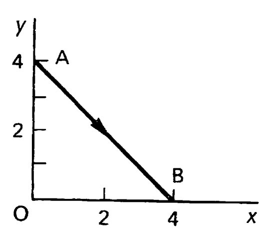
So far, we have concentrated on linear transformations of the form $w = az + b$. We can now proceed to something rather more interesting.

1 *Transformation* $w = z^2$ (refer to Frame 12)

The general principles are those we have used before. An example will show the development.

Example 1

The straight line AB in the z-plane as shown is mapped onto the w-plane by $w = z^2$.



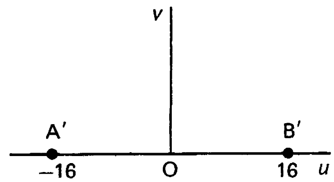
As before, we start by transforming the end points onto A′ and B′ in the w-plane.

A′: $w = \ldots\ldots\ldots\ldots$

B′: $w = \ldots\ldots\ldots\ldots$

33

$$\boxed{\text{A}': \ w = -16; \quad \text{B}': \ w = 16}$$



We cannot however assume that AB maps onto the straight line A′B′, since the transformation is not linear. We therefore have to deal with a general point.

$$w = u + jv = z^2 = (x + jy)^2 = x^2 + j2xy - y^2 = (x^2 - y^2) + j2xy$$

$$\therefore \ u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

The Cartesian equation of AB in the z-plane is $y = 4 - x$. So, substituting in the results of the previous line, we can express u and v in terms of x.

$$u = \dots\dots\dots\dots; \quad v = \dots\dots\dots\dots$$

34

$$\boxed{u = 8x - 16; \quad v = 8x - 2x^2}$$

The first gives $x = \dfrac{u + 16}{8}$ and substituting this in the expression for v gives . . . .

35

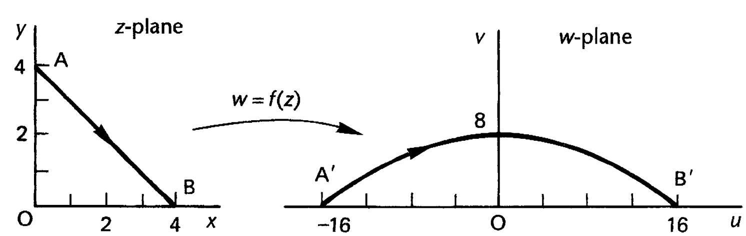
$$\boxed{v = -\tfrac{1}{32}u^2 + 8}$$

Because

$$v = 8\left(\frac{u + 16}{8}\right) - 2\left(\frac{u + 16}{8}\right)^2 = u + 16 - \frac{u^2}{32} - u - 8$$

$$\therefore \ v = -\frac{u^2}{32} + 8$$

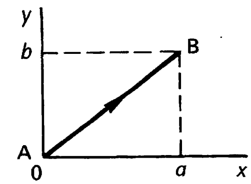
which is an 'inverted' parabola, symmetrical about the v-axis, with $v = 8$ at $u = 0$. The mapping is therefore



36

Example 2

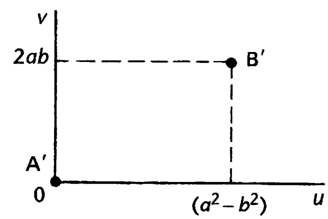
AB is a straight line in the z-plane joining the origin A to the point B $(a + jb)$. Obtain the mapping of AB onto the w-plane under the transformation $w = z^2$.



As always, first map the end points.

A′: $w = 0$

B′: $w = (a + jb)^2 = (a^2 - b^2) + j2ab$



Now to find the path joining A′ and B′, we consider a general point $z = x + jy$.

$$w = u + jv = z^2$$
$$= (x + jy)^2$$
$$= (x^2 - y^2) + j2xy$$
$$\therefore \; u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

The equation of AB is $y = \frac{b}{a}x$. We can therefore find u and v in terms of x and hence v in terms of u.

$u = \ldots\ldots\ldots\ldots$

$v = \ldots\ldots\ldots\ldots$

$v = f(u) = \ldots\ldots\ldots\ldots$

37

$$u = \left(\frac{a^2 - b^2}{a^2}\right)x^2; \quad v = \left(\frac{2b}{a}\right)x^2; \quad v = \left(\frac{2ab}{a^2 - b^2}\right)u$$

Because

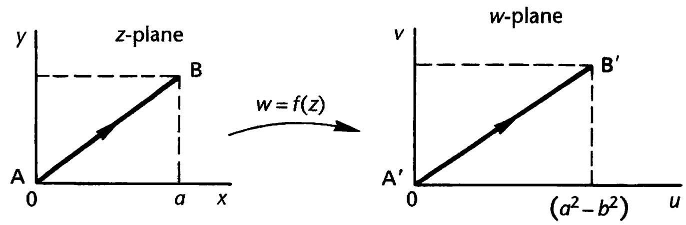
$$u = x^2 - y^2 = x^2 - \left(\frac{b^2}{a^2}\right)x^2 = \left(\frac{a^2 - b^2}{a^2}\right)x^2$$

$$v = 2xy = 2x\left(\frac{b}{a}\right)x = \left(\frac{2b}{a}\right)x^2$$

From the expression for u, $x^2 = \left(\frac{a^2}{a^2 - b^2}\right)u \quad \therefore \; v = \frac{2b}{a}\left(\frac{a^2}{a^2 - b^2}\right)u$

$\therefore \; v = \left(\frac{2ab}{a^2 - b^2}\right)u$ which is of the form $v = ku$.

A′B′ is therefore a straight line through the origin.



Therefore, under the transformation $w = z^2$, a straight line through the origin in the z-plane maps onto a straight line through the origin in the w-plane, whereas a straight line not passing through the origin maps onto a parabola.

This is worth remembering, so make a note of it

38

Example 3

A triangle consisting of AB, BC, CA in the z-plane is mapped onto the w-plane by the transformation $w = z^2$.

The transformation is $w = z^2$.

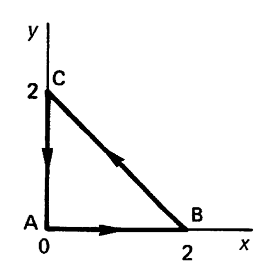
$$\therefore \; w = (x + jy)^2 = (x^2 - y^2) + j2xy$$
$$= u + jv$$
$$\therefore \; u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

First we can map the end points A, B, C onto A′, B′, C′ in the w-plane.

A′: ....

B′: ....

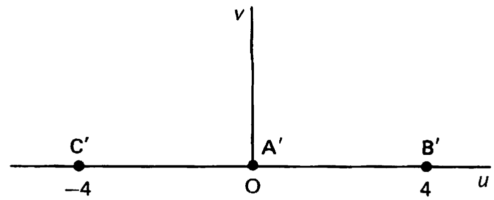
C′: ....



39

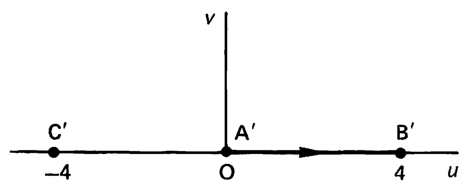
$$\boxed{A': \ w = 0; \quad B': \ w = 4; \quad C': \ w = -4}$$

So we establish



To find the paths joining these three transformed end points, we consider each of the sides of the triangle in turn.

(a) AB: Equation of AB is $y = 0 \quad \therefore \ u = x^2; \quad v = 0$
$\therefore$ Each point in AB maps onto a point between A′ and B′ for which $v = 0$, i.e. part of the u-axis.



(b) BC: Equation of BC is $y = 2 - x$
Substitute in $u = x^2 - y^2$ and $v = 2xy$ and determine v as a function of u.

$u = \ldots\ldots\ldots\ldots$
$v = \ldots\ldots\ldots\ldots$
$v = f(u) = \ldots\ldots\ldots\ldots$

40

$$\boxed{u = 4x - 4; \quad v = 4x - 2x^2; \quad v = 2 - \frac{u^2}{8}}$$

Because

$$u = x^2 - y^2 = x^2 - (2 - x)^2 = 4x - 4 \quad \therefore \ x = \frac{u + 4}{4}$$
$$v = 2xy = 2x(2 - x) = 4x - 2x^2$$
$$\therefore \ v = 4\left(\frac{u+4}{4}\right) - 2\left(\frac{u+4}{4}\right)^2 = 2 - \frac{u^2}{8}$$

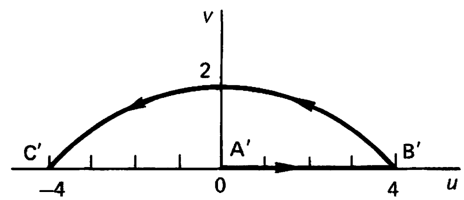
Therefore, the path joining B′ to C′ is an

$\ldots\ldots\ldots\ldots$

41

$\boxed{\text{inverted parabola}}$

$v = 2 - \dfrac{u^2}{8}$ $\quad \therefore$ at $u = 0$, $v = 2$ and the w-plane diagram now becomes



To complete the mapping, we have still to deal with CA. This transforms onto

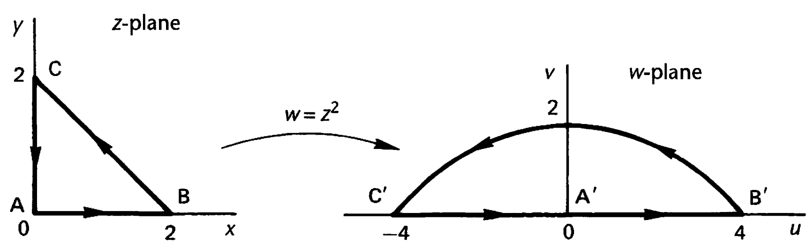
. . . .

42

$\boxed{\text{the } u\text{-axis between C}' \text{ and A}'}$

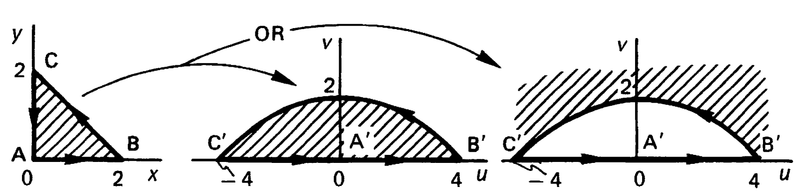
(c) CA: Equation of CA is $x = 0$ $\quad \therefore\ u = -y^2, \quad v = 0$
$\therefore$ Each point between C and A maps onto the negative part of the u-axis between C′ and A′.

So finally we have



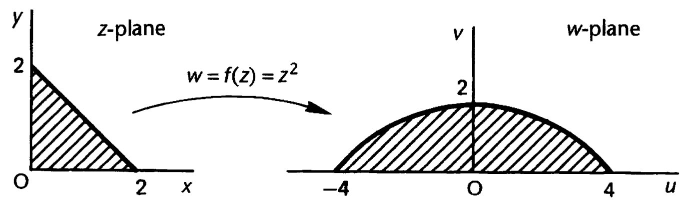
Mapping of regions

In this last example, the three lines AB, BC and CA form the boundary of a triangular region and we have seen how this boundary maps onto the boundary A′B′C′A′ in the w-plane. What we do not know yet is whether the points internal to the triangle map to points internal to the figure in the w-plane or to points external to it.



In the z-plane, the region is on the left-hand side as we proceed round the figure in the direction of the arrows ABCA. The region on the left-hand side as we proceed round the figure A′B′C′A′ in the w-plane determines that the transformed region in this case is, in fact, the internal region.

So



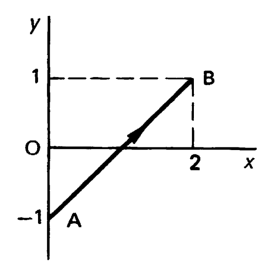
Therefore, every point in the region shaded in the z-plane maps onto a corresponding point in the region shaded in the w-plane.

43

2 *Transformation* $w = \frac{1}{z}$ (inversion)

Example 1

A straight line joining A $(-j)$ and B $(2+j)$ in the z-plane is mapped onto the w-plane by the transformation equation $w = \frac{1}{z}$.



Proceeding as before

$$w = \frac{1}{z}$$

$$\therefore\ u + jv = \frac{1}{x + jy}$$

$$= \frac{x - jy}{x^2 + y^2}$$

$$\therefore\ u = \frac{x}{x^2 + y^2}; \quad v = \frac{-y}{x^2 + y^2}$$

First we map the end points A and B onto the w-plane.

A′: $w = \ldots\ldots\ldots\ldots$

B′: $w = \ldots\ldots\ldots\ldots$

44

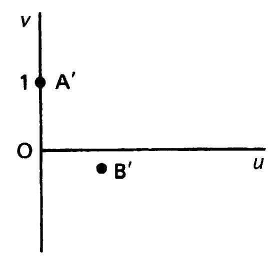
$$\boxed{A'\text{: } w = j; \quad B'\text{: } w = \frac{2}{5} - j\frac{1}{5}}$$

Because

A: $x = 0,\ y = -1$ $\quad\therefore$ A′: $u = 0,\ v = 1$ $\quad\therefore$ A′ is $w = j$

B: $x = 2,\ y = 1$ $\quad\therefore$ B′: $u = \frac{2}{5},\ v = -\frac{1}{5}$ $\quad\therefore$ B′ is $w = \frac{2}{5} - j\frac{1}{5}$

So far then we have



To determine the path A′B′, we can proceed as follows

$$w = \frac{1}{z} \qquad \therefore\ z = \frac{1}{w} \quad \text{i.e.} \quad x + jy = \frac{1}{u + jv} = \frac{u - jv}{u^2 + v^2}$$

$$\therefore\ x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

The equation of AB is $y = x - 1$

$$\therefore\ \frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1$$

which simplifies into ....

45

$$\boxed{u^2 + v^2 - u - v = 0}$$

Because

$$\frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1 \quad \therefore\ -v = u - u^2 - v^2$$

$$\therefore\ u^2 + v^2 - u - v = 0$$

We can write this as $(u^2 - u) + (v^2 - v) = 0$ and completing the square in each bracket this becomes

$$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2}$$

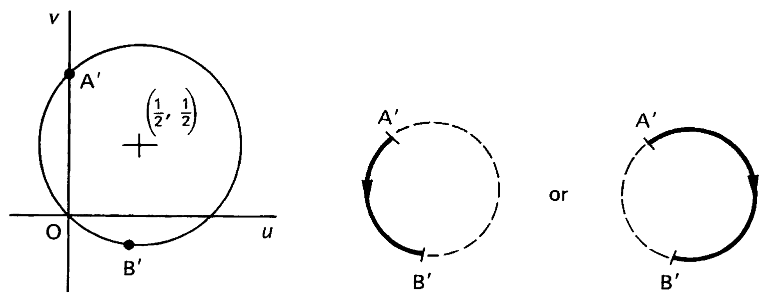
which we recognise as the equation of a ....

46

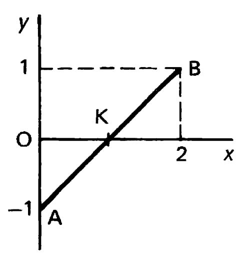
$$\boxed{\text{circle with centre } \left(\frac{1}{2}, \frac{1}{2}\right) \text{ and radius } \frac{1}{\sqrt{2}}}$$

The path joining A′ and B′ is therefore an arc of this circle.

But we still have problems, for it could be the minor arc or the major arc.



To decide which is correct, we take a further convenient point on the original line AB and determine its image on the w-plane.



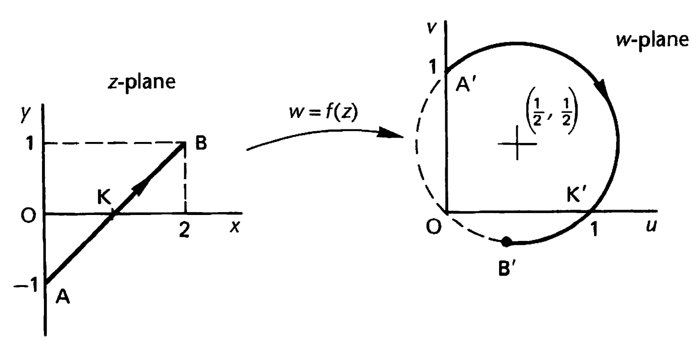
For instance, for K, $x = 1$, $y = 0$

$\therefore$ For K′, $u = \dfrac{x}{x^2 + y^2} = 1$

$v = \dfrac{-y}{x^2 + y^2} = 0$

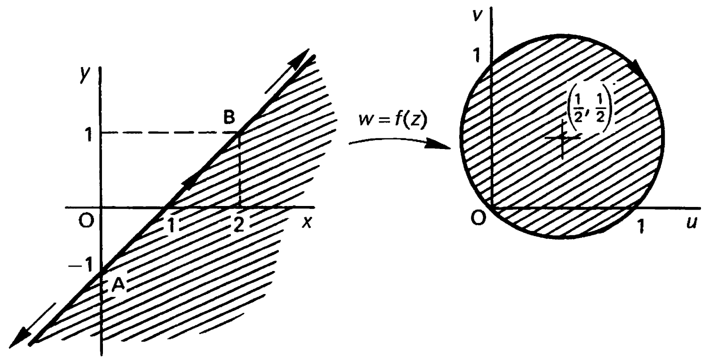
$\therefore$ K′ is the point $w = 1$

The path is, therefore, the major arc A′K′B′ developed in the direction indicated.



47

If we consider the line AB of the previous example extended to infinity in each direction, its image in the w-plane would then be the complete circle.



Furthermore, the line AB cuts the entire z-plane into two regions and

(a) the region on the right-hand side of the line relative to the arrowed direction maps onto the region inside the circle in the w-plane

(b) the region on the left-hand side of the line maps onto

....

48

the region outside the circle in the w-plane

Let us now consider a general case.

Example 2

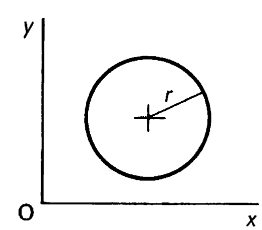
Determine the image in the w-plane of a circle in the z-plane under the inversion transformation $w = \dfrac{1}{z}$.

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with centre $(-g, -f)$

and radius $\sqrt{g^2 + f^2 - c}$.



It is convenient at times to write this as

$$A(x^2 + y^2) + Dx + Ey + F = 0$$

in which case

centre is .... and radius is ....

49

$$\boxed{\text{centre } \left(-\frac{D}{2A}, -\frac{E}{2A}\right); \quad \text{radius} = \frac{1}{2A}\sqrt{D^2 + E^2 - 4AF}}$$

Because

$$g = \frac{D}{2A}, \quad f = \frac{E}{2A}, \quad c = \frac{F}{A}.$$

As before we have $w = \dfrac{1}{z} \quad \therefore \; z = \dfrac{1}{w}$

$$\therefore \; x + jy = \frac{1}{u + jv} = \frac{u - jv}{u^2 + v^2} \quad \therefore \; x = \frac{u}{u^2 + v^2}; \quad y = \frac{-v}{u^2 + v^2}$$

Then $A(x^2 + y^2) + Dx + Ey + F = 0$

becomes ....

Simplify it as far as possible

50

$$\boxed{A + Du - Ev + F(u^2 + v^2) = 0}$$

Because we have

$$\frac{A(u^2 + v^2)}{(u^2 + v^2)^2} + \frac{Du}{u^2 + v^2} - \frac{Ev}{u^2 + v^2} + F = 0$$

$$\therefore \; A + Du - Ev + F(u^2 + v^2) = 0$$

Changing the order of terms, this can be written

$$F(u^2 + v^2) + Du - Ev + A = 0$$

which is the equation of a circle with

centre ....; radius ....

51

$$\boxed{\text{centre } \left(-\frac{D}{2F}, \frac{E}{2F}\right); \quad \text{radius} \frac{1}{2F}\sqrt{D^2 + E^2 - 4FA}}$$

Thus any circle in the z-plane transforms, with $w = \dfrac{1}{z}$, onto another circle in the w-plane.

We have already seen previously that, under inversion, a straight line also maps onto a circle. This may be regarded as a special case of the general result, if we accept a straight line as the circumference of a circle of .... radius.

52

infinite

Because

$$A(x^2 + y^2) + Dx + Ey + F = 0$$

If $A = 0$, this becomes $Dx + Ey + F = 0$ i.e. a straight line

and also the centre $\left(-\dfrac{D}{2A}, -\dfrac{E}{2A}\right)$ becomes infinite

and the radius $\dfrac{1}{2A}\sqrt{D^2 + E^2 - 4AF}$ becomes infinite.

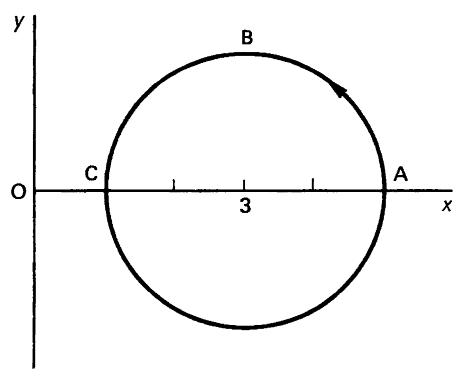
Therefore, combining the results we have obtained, we have this conclusion:

> Under inversion $w = \dfrac{1}{z}$, a circle or a straight line in the z-plane transforms onto a circle or a straight line in the w-plane.

Now for one more example.

Example 3

A circle in the z-plane has its centre at $z = 3$ and a radius of 2 units. Determine its image in the w-plane when transformed by $w = \dfrac{1}{z}$.



Equation of the circle is

$$(x - 3)^2 + y^2 = 4$$
$$x^2 - 6x + 9 + y^2 = 4$$
$$x^2 + y^2 - 6x + 5 = 0.$$

Using $w = \dfrac{1}{z}$, we can obtain x and y in terms of u and v.

$x = \dots\dots\dots\dots;\quad y = \dots\dots\dots\dots$

53

$$\boxed{x = \frac{u}{u^2+v^2}; \quad y = \frac{-v}{u^2+v^2}}$$

Because $w = \frac{1}{z}$,

$$\therefore \; z = \frac{1}{w}$$

$$\therefore \; x + jy = \frac{1}{u+jv}$$

$$= \frac{u - jv}{u^2+v^2}$$

$$\therefore \; x = \frac{u}{u^2+v^2}; \quad y = \frac{-v}{u^2+v^2}$$

Substituting these in the equation of the circle, we get a relationship between u and v, which is

....

54

$$\boxed{5(u^2+v^2) - 6u + 1 = 0}$$

Because the circle is $x^2 + y^2 - 6x + 5 = 0$

$$\therefore \; \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{6u}{u^2+v^2} + 5 = 0$$

$$\frac{1}{u^2+v^2} - \frac{6u}{u^2+v^2} + 5 = 0$$

$$5(u^2+v^2) - 6u + 1 = 0$$

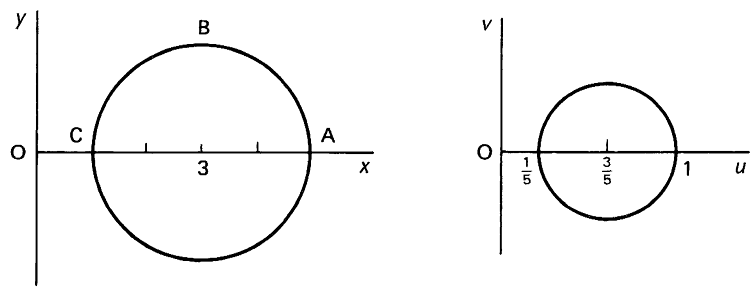
This is of the form $A(u^2+v^2) + Du + Ev + F = 0$
where $A = 5$, $D = -6$, $E = 0$, $F = 1$.

Therefore, the centre is ....
and the radius is ....

$$\text{centre} = \left(\frac{3}{5},\ 0\right); \quad \text{radius} = \frac{2}{5}$$

Because the centre is $\left(-\dfrac{D}{2A},\ -\dfrac{E}{2A}\right) = \left(\dfrac{6}{10},\ 0\right)$ i.e. $\left(\dfrac{3}{5},\ 0\right)$

and the radius $= \dfrac{1}{2A}\sqrt{D^2 + E^2 - 4AF} = \dfrac{1}{10}\sqrt{36 + 0 - 20} = \dfrac{2}{5}$.

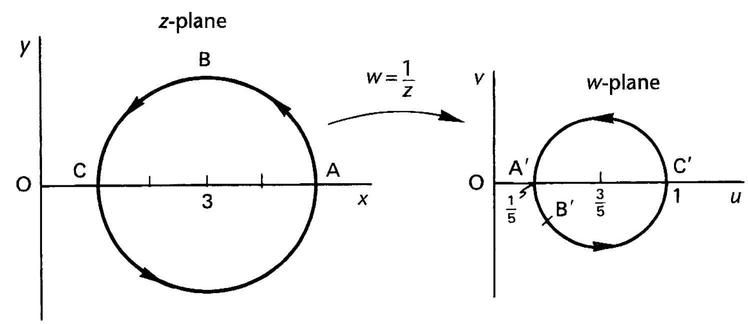


Taking three sample points A, B, C as shown, we can map these onto the w-plane using $u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$.

A′: ....; B′ : ....; C′: ....

$$\text{A}': \left(\frac{1}{5}, 0\right); \quad \text{B}': \left(\frac{3}{13},\ -\frac{2}{13}\right); \quad \text{C}': (1,\ 0)$$

So we finally have



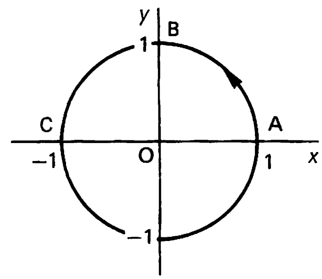
3 *Transformation* $w = \dfrac{1}{z - a}$

An extension of the method we have just applied occurs with transformations of the form $w = \dfrac{1}{z - a}$ where a is real or complex.

▶

Example

A circle $|z| = 1$ in the z-plane is mapped onto the w-plane by $w = \dfrac{1}{z-2}$.



$$w = \frac{1}{z-2} \quad \therefore \; z - 2 = \frac{1}{w}$$

$$x + jy - 2 = \frac{1}{u + jv}$$

$$(x-2) + jy = \frac{u - jv}{u^2 + v^2}$$

$$\therefore \; x = \frac{u}{u^2+v^2} + 2; \quad y = \frac{-v}{u^2+v^2}$$

Cartesian equation of the circle is $x^2 + y^2 = 1$.

We then substitute the expressions for x and y in terms of u and v and obtain the relationship between u and v, which is ....

57

$$\boxed{3(u^2+v^2) + 4u + 1 = 0}$$

Because we have $\left\{\dfrac{u + 2(u^2+v^2)}{u^2+v^2}\right\}^2 + \left\{\dfrac{-v}{u^2+v^2}\right\}^2 = 1$

$$\{u + 2(u^2+v^2)\}^2 + v^2 = (u^2+v^2)^2$$

$$u^2 + 4u(u^2+v^2) + 4(u^2+v^2)^2 + v^2 = (u^2+v^2)^2$$

$$1 + 4u + 4(u^2+v^2) = u^2 + v^2$$

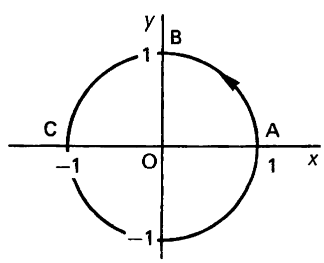
$$3(u^2+v^2) + 4u + 1 = 0$$

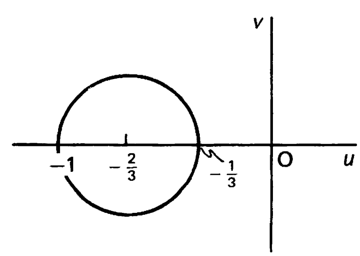
This can be expressed as

$$u^2 + \frac{4}{3}u + v^2 + \frac{1}{3} = 0$$

$$\left(u + \frac{2}{3}\right)^2 + v^2 = \left(\frac{1}{3}\right)^2$$

which is a circle with centre $\left(-\dfrac{2}{3},\, 0\right)$ and radius $\dfrac{1}{3}$.





To determine the direction of development relative to the arrowed direction in the z-plane, we consider the mapping of three sample points A, B, C as shown onto the w-plane, giving A′, B′, C′.

A′: ....; B′: ....; C′: ....

58

$$\boxed{A'\text{: } w=(-1,0);\quad B'\text{: } w=\left(-\frac{2}{5},-\frac{1}{5}\right);\quad C'\text{: } w=\left(\frac{1}{3},0\right)}$$

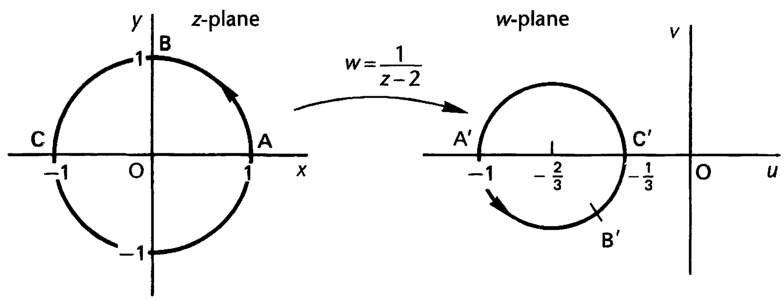
Because

$$\text{A: } z=1 \qquad \therefore\ w=\frac{1}{z-2}=-1 \qquad \therefore\ A'=(-1,0)$$

$$\text{B: } z=j \qquad \therefore\ w=\frac{1}{j-2}=\frac{j+2}{-5} \qquad \therefore\ B'=\left(-\frac{2}{5},-\frac{1}{5}\right)$$

$$\text{C: } z=-1 \qquad \therefore\ w=-\frac{1}{3} \qquad \therefore\ C'=\left(-\frac{1}{3},0\right)$$

Whereupon we have



We now have one further transformation which is important, so move on to the next frame for a fresh start

59

4 *Bilinear transformation* $w=\dfrac{az+b}{cz+d}$

Transformation of the form $w=\dfrac{az+b}{cz+d}$ where a, b, c, d are, in general, complex.

Note that

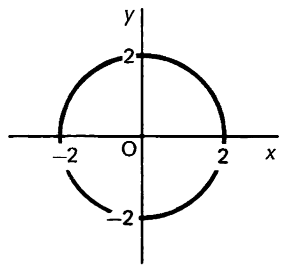
(a) if $cz+d=1$, $w=az+b$, i.e. the general linear transformation

(b) if $az+b=1$, $w=\dfrac{1}{cz+d}$, i.e. the form of inversion just considered.

▶

Example

Determine the image in the w-plane of the circle $|z| = 2$ in the z-plane under the transformation $w = \dfrac{z+j}{z-j}$ and show the region in the w-plane onto which the region within the circle is mapped.



We begin in very much the same way as before by expressing u and v in terms of x and y.

u ....; $v =$ ....

60

$$\boxed{u = \frac{x^2+y^2-1}{x^2+y^2-2y+1}; \quad v = \frac{2x}{x^2+y^2-2y+1}}$$

Because

$$\begin{aligned} w = u + jv = \frac{z+j}{z-j} &= \frac{x+j(y+1)}{x+j(y-1)} \\ &= \frac{\{x+j(y+1)\}\{x-j(y-1)\}}{\{x+j(y-1)\}\{x-j(y-1)\}} \\ &= \frac{x^2+jx(y+1-y+1)+y^2-1}{x^2+(y-1)^2} \\ &= \frac{x^2+y^2-1+j2x}{x^2+y^2-2y+1} \end{aligned}$$

$$\therefore \ u = \frac{x^2+y^2-1}{x^2+y^2-2y+1} \quad \text{and} \quad v = \frac{2x}{x^2+y^2-2y+1}$$

But the equation of the circle is $x^2 + y^2 = 4$, so these expressions simplify to

$u =$ .... and $v =$ ....

61

$$\boxed{u = \frac{3}{5-2y}; \quad v = \frac{2x}{5-2y}}$$

From these, we can form expressions for x and y in terms of u and v.

$x =$ ....; $y =$ ....

62

$$\boxed{x = \frac{3v}{2u}; \quad y = \frac{5u-3}{2u}}$$

Because, from the first, $y = \dfrac{5u-3}{2u}$ and substituting in the second gives $x = \dfrac{3v}{2u}$.

But $x^2 + y^2 = 4 \quad \therefore \quad \dfrac{9v^2}{4u^2} + \dfrac{(5u-3)^2}{4u^2} = 4$

which can be simplified to ....

63

$$\boxed{9(u^2 + v^2) - 30u + 9 = 0}$$

Because

$$9v^2 + 25u^2 - 30u + 9 = 16u^2 \quad \therefore \quad 9(u^2 + v^2) - 30u + 9 = 0.$$

Dividing through by 9, we can now rearrange this to

$$\left(u^2 - \frac{30}{9}u\right) + v^2 + 1 = 0$$

i.e. $$\left(u - \frac{5}{3}\right)^2 + v^2 + 1 - \frac{25}{9} = 0$$

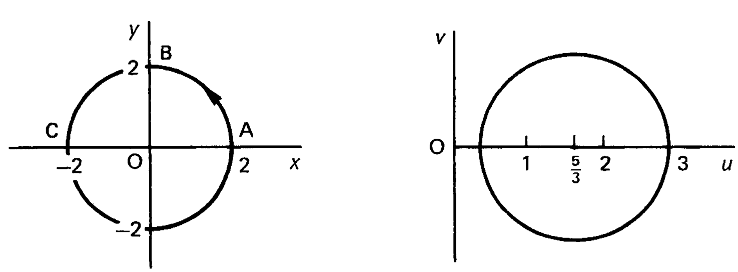
$$\left(u - \frac{5}{3}\right)^2 + v^2 = \left(\frac{4}{3}\right)^2$$

which, you will recognise, is a circle in the w-plane with

centre .... and radius ....

64

$$\boxed{\text{centre} = \left(\frac{5}{3}, 0\right); \quad \text{radius} = \frac{4}{3}}$$



To find the direction of development, we map three sample points A, B, C onto A′, B′, C′ as usual.

A′: ....; B′: ....; C′: ....

65

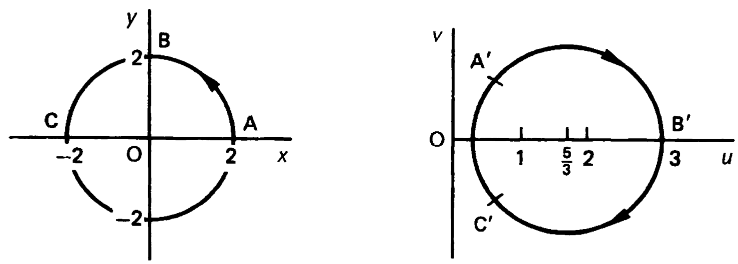
$$\boxed{A'\colon w = \frac{3}{5} + j\frac{4}{5};\quad B'\colon w = 3;\quad C'\colon w = \frac{3}{5} - j\frac{4}{5}}$$

Because

A: $z = 2 \qquad \therefore\ w = \dfrac{2+j}{2-j} = \dfrac{(2+j)^2}{5} = \dfrac{4+j4-1}{5} = \frac{3}{5} + j\frac{4}{5}$ i.e. A′

B: $z = j2 \qquad \therefore\ w = \dfrac{j2+j}{j2-j} = \dfrac{j3}{j} = 3 \quad \therefore\ w = 3$ i.e. B′

C: $z = -2 \qquad \therefore\ w = \dfrac{-2+j}{-2-j} = \dfrac{2-j}{2+j} = \dfrac{(2-j)^2}{5} = \frac{3}{5} - j\frac{4}{5}$ i.e. C′



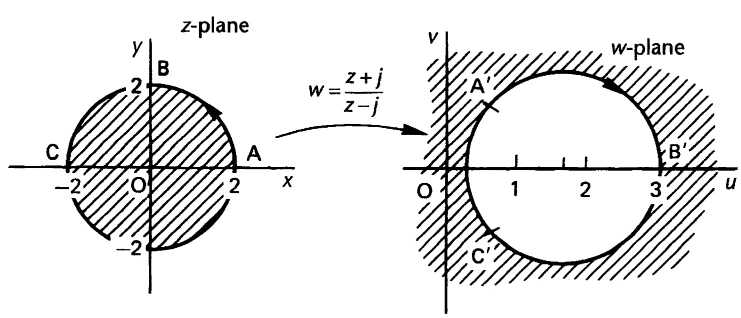
So an anticlockwise progression in the z-plane becomes a clockwise progression in the w-plane with this particular example.

Now we can complete the problem, for the region inside the circle in the z-plane maps onto .... in the w-plane.

66

the region outside the circle

Because the enclosed region in the z-plane is on the left-hand side of the direction of progression. The region on the left-hand side of the direction of progression in the w-plane is thus the region outside the transformed circle.



▶

And that brings us successfully to the end of this Programme. We shall pursue the topic further in the succeeding Programe. Meanwhile, all that remains is to check down the **Revision summary** and the **Can You?** checklist before working through the **Test exercise**. All very straightforward. The **Further problems** will give you valuable additional practice.

Revision summary 20

67

1 *Transformation equation*

$$z = x + jy \qquad w = u + jv$$

The transformation equation is the relationship between z and w, i.e. $w = f(z)$.

2 *Linear transformation* $w = az + b$ where a and b are real or complex. A straight line in the z-plane maps onto a corresponding straight line in the w-plane.

3 *Types of transformation* $w = az + b$

(a) *magnification* – given by $|a|$

(b) *rotation* – given by arg a

(c) *translation* – given by b.

4 *Non-linear transformation*

(a) $w = z^2$

A straight line through the origin maps onto a corresponding straight line through the origin in the w-plane. A straight line not passing through the origin maps onto a parabola.

(b) $w = \dfrac{1}{z}$ (inversion)

A straight line or a circle maps onto a straight line or a circle in the w-plane.

A straight line may be regarded as a circle of infinite radius.

(c) $w = \dfrac{az + b}{cz + d}$ (bilinear transformation) – with a, b, c, d real or complex.

5 *Mapping of a region* depends on the direction of development. Right-hand regions map onto right-hand regions: left-hand regions onto left-hand regions.

Can You?

68

Checklist 20

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: — Frames

- Recognise the transformation equation in the form $w = f(z) = u(x, y) + jv(x, y)$? — 1 and 2
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Illustrate the image of a point in the complex z-plane under a complex mapping onto the w-plane? — 2 to 7
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Map a straight line in the z-plane onto the w-plane under the transformation $w = f(z)$? — 7 to 16
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Identify complex mappings that form translations, magnifications, rotations and their combinations? — 16 to 31
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Deal with the non-linear transformations $w = z^2$, $w = 1/z$, $w = 1/(z-a)$ and $w = (az+b)/(cz+d)$? — 32 to 66
  Yes ☐ ☐ ☐ ☐ ☐ *No*

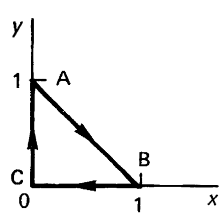
Test exercise 20

69

1 Map the following points in the z-plane onto the w-plane under the transformation $w = f(z)$.
(a) $z = 3 + j2$; $w = 2z - j6$
(b) $z = -2 + j$; $w = 4 + jz$
(c) $z = j(1 - j)$; $w = (2 + j)z - 1$
(d) $z = j - 2$; $w = (1 - j)(z + 3)$.

2 Map the straight line joining A $(2 - j)$ and B $(4 - j3)$ in the z-plane onto the w-plane using the transformation $w = (1 + j2)z + 1 - j3$. State the magnification, rotation and translation involved.

3 A triangle ABC in the z-plane as shown is mapped onto the w-plane under the transformation $w = z^2$.



Determine the image in the w-plane and indicate the mapping of the interior triangular region ABC.

4 Map the straight line joining A ($z = j$) and B ($z = 3 + j4$) in the z-plane onto the w-plane under the inversion transformation $w = \dfrac{1}{z}$.
Sketch the image of AB in the w-plane.

5 The unit circle $|z| = 1$ in the z-plane is mapped onto the w-plane by $w = \dfrac{1}{z - j2}$. Determine (a) the position of the centre and (b) the radius of the circle obtained.

6 The circle $|z| = 2$ is mapped onto the w-plane by the transformation $w = \dfrac{z + j2}{z + j}$. Determine the centre and radius of the resulting circle in the w-plane.

Further problems 20

70

1 A triangle ABC in the z-plane with vertices A $(-1 - j)$, B $(2 + j2)$, C $(-1 + j2)$ is mapped onto the w-plane under the transformation $w = (1 - j)z + (1 + j2)$. Determine the image A′B′C′ of ABC in the w-plane.

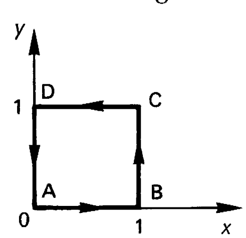
2 The straight line joining A $(1 + j2)$ and B $(4 - j3)$ in the z-plane is mapped onto the w-plane by the transformation equation $w = (2 + j5)z$. Determine (a) the images of A and B, (b) the magnification, rotation and translation involved.

3 Map the straight line joining A $(-2 + j3)$ and B $(1 + j2)$ in the z-plane onto the w-plane using the transformation equation

$$w = (-3 + j)z + 2 + j4.$$

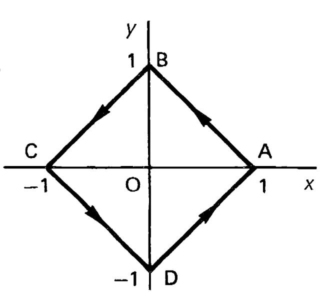
State the magnification, rotation and translation occurring in the process.

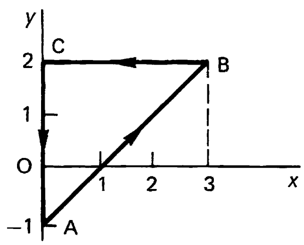
4



Transform the square ABCD in the z-plane onto the w-plane under the transformation $w = z^2$.

5 Map the square ABCD in the z-plane onto the w-plane using the transformation $w = 2z^2 + 2$.



6 

The triangle ABC in the z-plane is mapped onto the w-plane by the transformation $w = j2z^2 + 1$. Determine the image of ABC in the w-plane.

7 A circle in the z-plane has its centre at the point $(-\frac{3}{4} - j)$ and radius $\frac{7}{4}$. Show that its Cartesian equation can be expressed as

$$2(x^2 + y^2) + 3x + 4y - 3 = 0$$

Determine the image of the circle in the w-plane under the inversion transformation $w = \dfrac{1}{z}$.

8 The transformation $w = \dfrac{1}{z-1}$ is applied to the circle $|z| = 2$ in the z-plane. Determine

(a) the image of the circle in the w-plane

(b) the region in the w-plane onto which the region enclosed within the circle in the z-plane is mapped.

9 The circle $|z| = 4$ is described in the z-plane in an anticlockwise manner. Obtain its image in the w-plane under the transformation $w = \dfrac{z+1}{z-2}$ and state the direction of development.

10 The bilinear transformation $w = \dfrac{z-j}{z+j2}$ is applied to the circle $|z| = 3$ in the z-plane. Determine the equation of the image in the w-plane and state its centre and radius.

11 The unit circle $|z| = 1$ in the z-plane is mapped onto the w-plane under the transformation $w = \dfrac{z-1}{z-3}$. Determine the equation of its image and the region onto which the region within the circle is mapped.

12 Obtain the image of the unit circle $|z| = 1$ in the z-plane under the transformation $w = \dfrac{z+j3}{z-j2}$.

13 The circle $|z| = 2$ is mapped onto the w-plane by the transformation $w = \dfrac{z+j}{2z-j}$. Determine

(a) the image of the circle in the w-plane

(b) the mapping of the region enclosed by $|z| = 2$.

14 Show that the transformation equation $w = \dfrac{z-a}{z-b}$ where $z = x + jy$, $a = 1 + j4$ and $b = 2 + j3$, transforms the circle $(x-3)^2 + (y-5)^2 = 5$ into a straight line through the origin in the w-plane.